𝕋–Orbit Spaces of Multiplicative Actions and Applications

Evelyne Hubert, Tobias Metzlaff*, Philippe Moustrou, Cordian Riener
*tobias.metzlaff@inria.fr

(1) Synopsis

- The set of all orbits of a compact Lie group $\mathcal{G}$ with an action on an affine variety is called **orbit space**.
- In the case of Weyl groups, which fix a lattice Ω, the orbit space of $\mathcal{G}$ as the image of the compact torus $\mathbb{T}^n$ under fundamental invariants is a compact basic semi–algebraic set.
- The coordinate ring $\mathbb{Q}[x^\pm]$ of the affine variety $(\mathbb{C}^*)^n$ contains the ring of invariants $\mathbb{Q}[x^\pm]^{\mathcal{G}}$.
- The orbit space is characterized as the positivity–locus of an explicitly given matrix polynomial H.
- Application: Bounds for chromatic numbers of infinite graphs via polynomial optimization.

(2) Multiplicative Actions and Invariants

The nonlinear action

$$\begin{array}{rccl} \star: & \mathcal{G}\times(\mathbb{C}^*)^n & \to & (\mathbb{C}^*)^n, \\ & (B,x) & \mapsto & (x^{B^{-1}_{\cdot 1}},\dots,x^{B^{-1}_{\cdot n}}), \end{array}$$

leaves the compact torus $\mathbb{T}^n$ invariant. The Laurent polynomials, which are invariant under the induced **multiplicative** action of $\mathcal{G}$ on $\mathbb{Q}[x^\pm]$, form a polynomial algebra

$$\mathbb{Q}[x^\pm]^{\mathcal{G}} = \mathbb{Q}[\theta_1(x),\dots,\theta_n(x)],$$

where $\theta_i = 1/|\mathcal{G}|\sum_{B\in\mathcal{G}} x^{B_{\cdot i}}$ are fundamental invariants.

Goal: Describe the orbit space.

Fact: The image of the fundamental invariants corresponds to the set of orbits of $\mathbb{T}^n$. We call it the $\mathbb{T}$**–orbit space of** $\mathcal{G}$ and denote it $\mathcal{T}$.

(3) Main Result for A_{n-1}, B_n, C_n, D_n

For $z\in\mathbb{R}^n$, let $f(t)$ be the monic polynomial of degree n with coefficients $(-1)^i z_i$. The multiplication by t in $\mathbb{R}[t]/f$ is given by a polynomial matrix $C(z)$. Define the **Hermite matrix polynomial** $H(z)\in\mathbb{Q}[z]^{n\times n}$ with entries

$$H(z)_{ij} = \mathrm{Trace}(4\,C(z)^{i+j-2} - C(z)^{i+j}).$$

For all $z\in\mathbb{R}^n$, $H(z)$ is real symmetric and the $\mathbb{T}$–orbit space of $\mathcal{G}$ is $\mathcal{T} = \{z\in\mathbb{R}^n \mid H(z)\succeq 0\}$.

(4) Example

Consider the Weyl group $\mathfrak{S}_2\ltimes\{\pm1\}^2$ of the C_2 root system (permutation and sign change of coordinates in $\mathbb{R}^2$). In the basis of fundamental weights, the generators of the integer representation $\mathcal{G}$ are

$$\begin{pmatrix}-1 & 0\\ 2 & 1\end{pmatrix}\quad\begin{pmatrix}1 & 1\\ 0 & -1\end{pmatrix}$$

and, for σ_i the elementary symmetric polynomials, the fundamental invariants are

$$\theta_i(x) = 1/4\,\sigma_i(x_1+x_1^{-1}, x_2^2x_1^{-1}+x_2^{-2}x_1)\in\mathbb{Q}[x^\pm]^{\mathcal{G}},\quad i=1,2.$$

Let $z\in\mathbb{R}^2$. By Vieta's formula, $f(t):=t^2-4\,z_1\,t+4\,z_2$ has all of its roots in $[-2,2]$ if and only if z has a preimage in $\mathbb{T}^2$. Hence, the $\mathbb{T}$–orbit space is the positivity locus of the associated Hermite quadratic form

$$H(z) = \begin{pmatrix} -2z_1^2+z_2+1 & -8z_1^3+6z_1z_2+2z_1 \\ -8z_1^3+6z_1z_2+2z_1 & -32z_1^4+8z_1^2+32z_1^2z_2-4z_2^2-4z_2\end{pmatrix}.$$

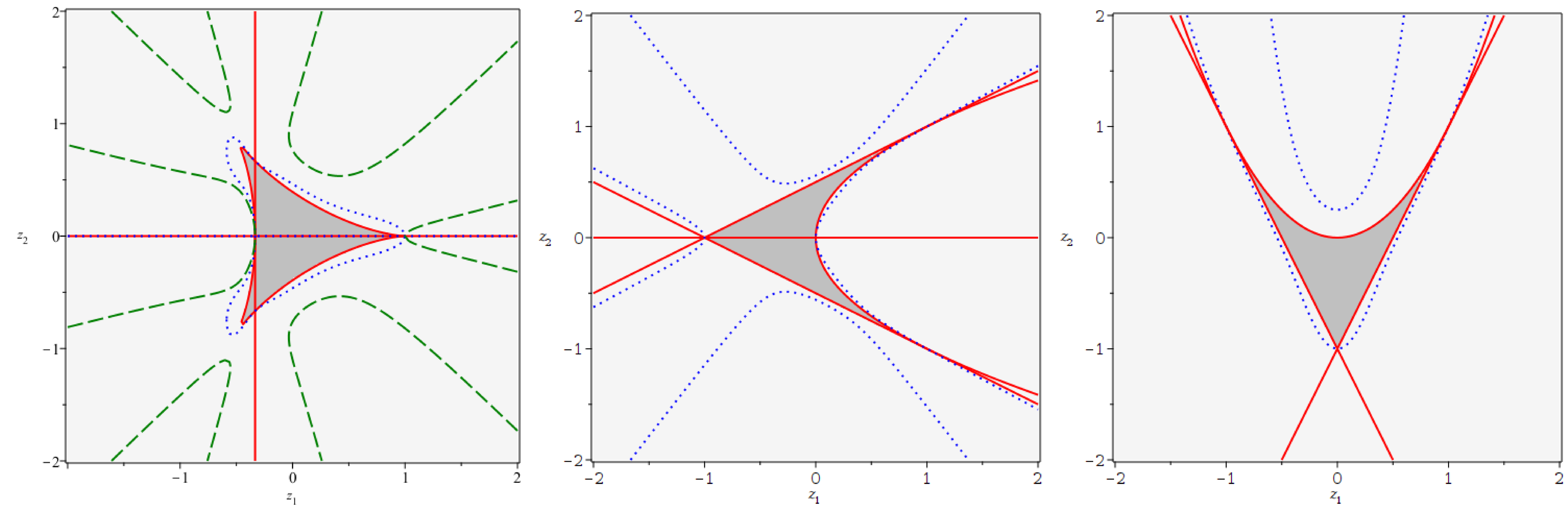


Figure 1: $\mathbb{T}$–orbit spaces for the Weyl groups of root systems A_2 (left), B_2 (middle), C_2 (right).

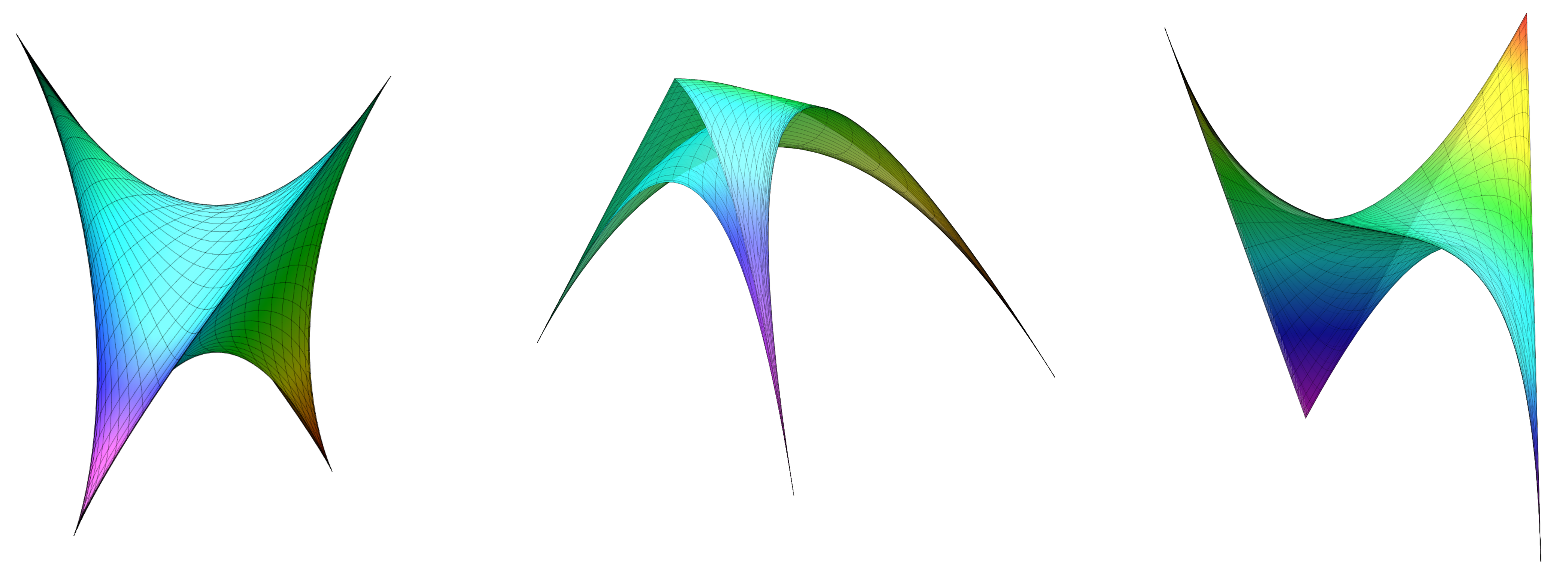


Figure 2: $\mathbb{T}$–orbit spaces for the Weyl groups of root systems A_3 (left), B_3 (middle), C_3 (right).

(5) Optimization of Trigonometric Polynomials

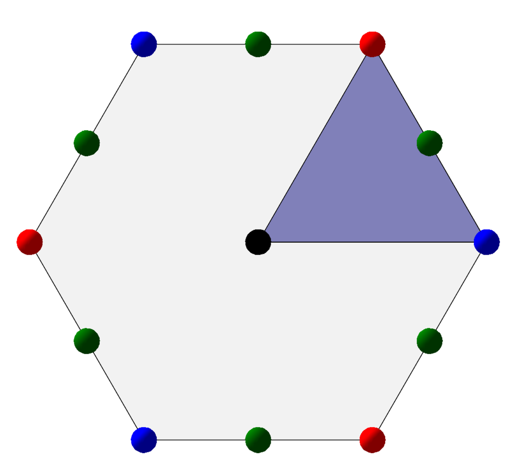
For a sequence $(f_\mu)_{\mu\in\Omega}\subseteq\mathbb{Q}$ with finite support, a **trigonometric polynomial** is a map

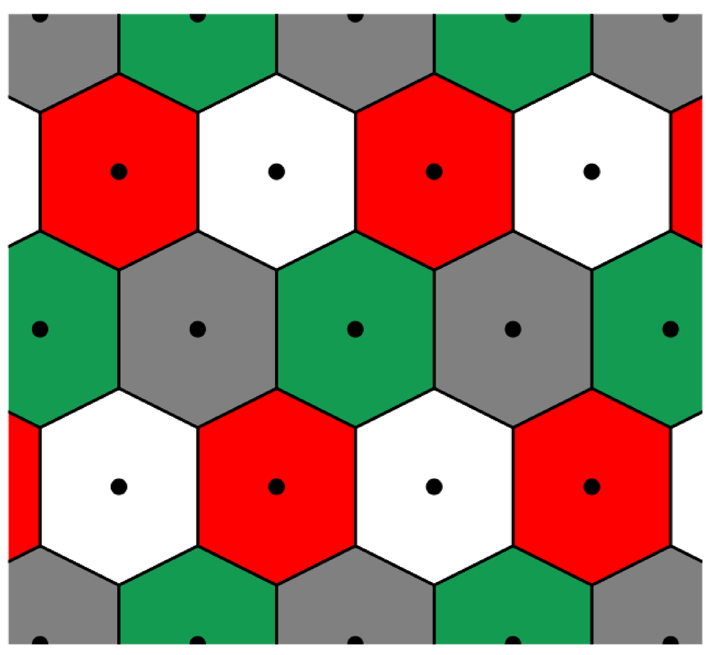
$$\begin{array}{rccl} f: & \mathbb{R}^n & \to & \mathbb{C}, \\ & u & \mapsto & \sum\limits_{\mu\in\Omega} f_\mu\,\exp(-2\pi\mathrm{i}\,\langle\mu,u\rangle). \end{array}$$

Under the assumption that f is real–valued and invariant under $\mathcal{G}$, the problem of determining the optimal value of f on $\mathbb{R}^n$ can be written as a polynomial optimization problem

$$f^* = \min_{u\in\mathbb{R}^n} f(u) = \min_{z\in\mathcal{T}} \sum_{\alpha\in\mathbb{N}^n} v_\alpha\,T_\alpha(z),$$

where the T_α are generalized Chebyshev polynomials. The value f^* arises for example in the computation of chromatic numbers for infinite graphs [1]. Knowing that $\mathcal{T}$ is the positivity locus of the Hermite matrix polynomial, we follow the idea of [3] and relax to a semi–definite program.





Maple Package: https://www-sop.inria.fr/members/Tobias.Metzlaff/GeneralizedChebyshev.zip

Further Information ...

References

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